Engineering Notes

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Axisymmetric Vibrations of a Fluid Filled Cylindrical Shell Submerged in a Heavy Fluid

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Nomenclature

a = outer radius of the shell
A = axial amplitude
c_o = speed of sound in the surrounding liquid
C = radial amplitude
E = Young's modulus of elasticity
h = shell thickness
i = subscript referring to the internal fluid
k = eigenvalue
K_i = bulk modulus of elasticity of the internal fluid
L = length of the shell
m_L = axially excited added-mass term, $8\rho_o a/3\pi$
m_r = radially excited added-mass term, $2\rho_o c_o^2/\omega^2 a$
$N\varphi$ = pressure-induced circumferential stress term, $(p_i - p_o)a$
N_x = pressure-induced axial stress term, $\frac{1}{2}N\varphi$
o = subscript referring to surrounding fluid
p = pressure
t = time
u = axial displacement
w = radial displacement
x = axial coordinate
$\gamma(kL)$ = $[\sinh(kL) - \sin(kL)]/[\cosh(kL) - \cos(kL)]$
ν = Poisson's ratio
ρ = mass density
ω = circular frequency

I. Introduction

THE analysis of the vibratory motions of a line array housing is complicated by both the frequency dependent added mass of the internal and surrounding fluids and by the internal heterogeneous structure of the array. The arrays consist of hydrophones mounted in various manners within the housing. These hydrophones are surrounded by a liquid that transmits signals and insures neutral buoyancy of the array.

One of the most significant studies of the vibrational characteristics of fluid-filled cylindrical shells was performed by Berry and Reissner.[1] In that study the effect of an internal compressible fluid on the breathing modes of a cylindrical shell was assumed to be an added-mass effect, i.e., the fluctuating internal pressure was assumed to be equal to the inertial reaction of the internal fluid. The method of analysis in Ref. 1 is that used by Reissner in an earlier paper.[2] In Ref. 2 the axial inertia of the shell is assumed to be negligible, thus, allowing the use of Airy's stress function in the analysis.

The effect of a dense surrounding fluid on the vibrations of a periodically supported circular shell of infinite length was studied by Herman and Klosner.[3] The analysis of Ref. 3 utilizes the uncoupled Timoshenko shell equations neglecting the axial inertia and the axial stress. The surrounding fluid is considered to be an acoustic media, and therefore, the added mass is frequency-dependent.

In the present study a variational analysis of the free vibrations of the coupled axial and radial modes of a liquid-filled cylindrical shell submerged in a dense fluid is performed. The shell is of finite length and is assumed to be clamped at both ends in the radial mode, while the axial mode is clamped at one end and free at the other end. The internal and external fluids are assumed to be compressible. The influence of the pressure on the natural frequencies of the shell is also studied.

2. Equations of Motion

The equations of motion describing the axisymmetric modes of a liquid filled circular cylindrical shell submerged in a dense fluid are the following:

$$L_1(u,w) - [(1 - \nu^2)/Eh](m_L + \rho h)(\partial^2 u/\partial t^2) = 0 \quad (1)$$

where

$$L_1(u,w) = (\partial^2 u/\partial x^2) - (\nu/a)(\partial w/\partial x) + N\varphi[(1 - \nu^2)/Eha](\partial w/\partial x)$$

and

$$L_2(u,w) - \frac{a(1 - \nu^2)}{Eh}(m_r + \rho h)(\partial^2 w/\partial t^2) = 0 \quad (2)$$

where

$$L_2(u,w) = \nu\frac{\partial u}{\partial x} - \left[\frac{-K_i}{E(a - h/2)} + \frac{1}{a}\right]w - \frac{ah^2}{12}\frac{\partial^4 w}{\partial x^4} + \frac{a(1 - \nu^2)}{Eh}N_x\frac{\partial^2 w}{\partial x^2}$$

The internal compressible fluid can be represented in two ways. First, if the inertial reaction of the fluid is neglected, the term $-K_i/[E(a - h/2)]$ is used in Eq. (2). If the inertial reaction of the internal added mass is considered, then the term m_r can be modified to include this added mass as in Ref. 1.

The displacements in the respective axial and radial directions are assumed to have the following forms:

$$u = [A\cos(kx) + B\cosh(kx)]\cos(\omega t) \quad (3)$$

and

$$w = [C\sin(kx) + D\cos(kx) + E\sinh(kx) + F\cosh(kx)]\cdot\cos(\omega t) \quad (4)$$

where k is an eigenvalue and ω is the circular frequency. Equations (3) and (4) are assumed to satisfy the following boundary conditions:

$$u = w = dw/dx = 0 \text{ at } x = 0$$

and $w = dw/dx = 0$ at $x = L$. Thus, Eqs. (3) and (4) are, respectively,

$$u = A[\cos(kx) - \cosh(kx)]\cos(\omega t) \quad (5)$$

and

$$w = C\{-\sinh(kx) + \sin(kx) - \gamma(kL)[\cos(kx) - \cosh(kx)]\}\cos(\omega t) \quad (6)$$

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Table 1

Case	a/h	L/a	K_i/E	ρ_o/ρ	$(\Delta p)a/Eh$ $(\times 10^5)$	$\omega_n a/c_o$ $(\times 10^4)$ $n = 1$	2	3	4	5	6	7
1	17.3	278	39.0	0.667	250	6.79	11.48	16.1	20.7	25.2	29.8	34.1
2	17.3	278	0.00239	0.667	250	6.79	11.48	16.1	20.7	25.2	29.8	34.1
3	17.3	167	39.0	0.667	250	11.3	19.1	26.8	34.4	42.1	49.7	56.9
4	34.61	278	39.0	0.667	500	4.91	8.31	11.6	15.0	18.3	21.6	24.7
5	17.3	278	39.0	1.33	250	4.91	8.31	11.6	15.0	18.3	21.6	24.7
6	17.3	278	39.0	1.00	250	6.89	11.7	16.3	21.0	25.6	30.3	34.7
7	17.3	278	39.0	0.667	3233	6.79	11.5	16.1	20.7	25.2	29.8	34.1
8	173	278	39.0	0.667	2500	2.22	3.81	5.32	6.78	8.33	9.84	11.3
9	173	278	39.0	0.667	32330	2.28	3.81	5.33	6.82	8.34	9.86	11.4
10	173	278	39.0	1.33	2500	1.59	2.70	3.79	4.85	5.91	6.99	7.99
11	17.3	278	35.2	0.667	256	7.14	12.1	16.9	21.8	26.6	31.4	35.9

3. Natural Frequency

To determine the expression for the natural frequency, the variational method as described in Ref. 4, is used. Combining the displacement functions in Eqs. (5) and (6) with Eqs. (1) and (2), the variational equation can be written as

$$\int_0^L \int_{t_1}^{t_2} \left\{ \left[L_1(u,w) + \frac{a\omega^2}{\xi}(m_L + \rho h)u \right] \delta u + \left[L_2(u,w) + \frac{a\omega^2}{\xi}(m_r + \rho h)w \right] \delta w \right\} dxdt = 0 \quad (7)$$

where $\xi = Eh/(1 - \nu^2)$ and where the variations in the displacements are

$$\delta u = \delta A [\cos(kx) - \cosh(kx)] \cos(\omega t) \quad (8)$$

and

$$\delta w = \delta C \{\sin(kx) - \sinh(kx) - \gamma(kL)[\cos(kx) - \cosh(kx)]\} \cos(\omega t) \quad (9)$$

Since δA and δC are arbitrarily chosen, their coefficients in the resulting integration of Eq. (7) must be identically zero. Equating these coefficients to zero, the following relations are obtained:

$$A\{-[k + (m_L + \rho h)\omega^2/k\xi]T_1 + kT_2\} + C\{P_1[T_3\gamma(kL)/2k - T_1/k]\} = 0 \quad (10)$$

and

$$A\{-\nu(T_6 + T_7)\} + C\{\gamma(kL)[P_2 + a\rho h\omega^2/\xi]T_4/k + \gamma(kL)akN_xT_5/\xi - [P_2 + a\rho h\omega^2/\xi]T_6/k - akN_xT_7/\xi\} = 0 \quad (11)$$

where

$$T_1 = -kL + 1 + \sin(kL)\cosh(kL) - \tfrac{1}{2}\cosh^2(kL)$$

$$T_2 = -kL + 1 + \sin(kL)\cosh(kL)$$

$$T_3 = 1 - \sinh^2(kL)$$

$$T_4 = -\tfrac{1}{2} - \tfrac{1}{2}\sinh^2(kL) + \sin(kL)\sinh(kL) + \gamma(kL)[kL - 1 - \sin(kL)\cosh(kL) + \tfrac{1}{2}\cosh^2(kL)]$$

$$T_5 = 1 - \sin(kL)\sinh(kL) - \gamma(kL)[kL - 1 - \sin(kL)\cosh(kL)]$$

$$T_6 = -1 + \sin(kL)\cosh(kL) - \tfrac{1}{2}\cosh^2(kL) + \gamma(kL)[\tfrac{1}{2} - \sin(kL)\sinh(kL) + \tfrac{1}{2}\sinh^2(kL)]$$

$$T_7 = kL + 1 - \sin(kL)\cosh(kL) + \gamma(kL) \times [-1 + \sin(kL)\sinh(kL)]$$

$$P_1 = (N\varphi/\xi - \nu)k/a$$

and

$$P_2 = -\left\{-K_i\Big/\left[E\left(a - \frac{h}{2}\right)\right] + 1/a + ah^2k^4/12\right\} + k^2aN_x/\xi + 2\rho_o c_o^2/\xi$$

Since A and C are arbitrarily chosen, the determinant of their coefficients must be zero. Expansion of the determinant results in the frequency equation

$$\{-T_1a\rho h/(k\xi)^2(m_L + \rho h)[\gamma(kL)T_4 - T_6]\}\omega^4 + ([T_4\gamma(kL) - T_6]a\rho h(T_2 - T_1)/\xi - T_1(m_L + \rho h)/(k\xi)[P_2\{\gamma(kL)T_4 - T_6\}/k + akN_x\{\gamma(kL)T_5 - T_7\}/\xi])\omega^2 - k(T_2 - T_1)\{P_2[\gamma(kL)T_4 - T_6]/k + akN_x[\gamma(kL)T_5 - T_7]/\xi\} + \nu(T_6 + T_7)P_1[\gamma(kL)T_3/2 - T_1]/k = 0 \quad (12)$$

4. Results

The numerical results obtained from the theory are in Table 1. The values of the parameters used in case 1 of the table are the following: $\nu = 0.5$, $c_o = 4800$ fps, $a = 0.18$ ft, $L = 50$ ft, $h = 0.0104$ ft, $E = 9.22 \times 10^5$ lb/ft^2, $K_i = 36 \times 10^6$ lb/ft^2, $N_x = 12$ lb/ft, $\rho_o = 2.0$ slugs/ft^3, $N\varphi = 24$ lb/ft, $\rho = 3.0$ slugs/ft^3. Cases 2–11 were obtained by varying these values as follows: case 2, $K_i = 2200$; case 3, $L = 30$; case 4, $h = 0.0052$; case 5, $\rho_o = 4$; case 6, $\rho = 2$; case 7, $N\varphi = 310$; case 8, $h = 0.00104$; case 9, $h = 0.00104$ and $N\varphi = 310$; case 10, $h = 0.00104$ and $\rho_o = 4$; case 11, $E = 10.22 \times 10^5$.

5. Discussion and Conclusions

From the numerical results of the last section, it appears that the effect of the internal fluid is primarily an added mass effect. This can be seen by comparing cases 1 and 2 and cases 1 and 5. The variation in the bulk modulus appears to have no effect on the vibrational characteristics of the shell; however, by changing the mass density of the internal fluid (accomplished by altering m_r)[5] a significant change in the Strouhal number $\omega_n a/c_9$ results. Variation in the internal or ambient pressure does not affect the characteristics of the thicker shell as seen in cases 1–7 and in case 11. For the thin shell, cases 8–10, the pressure change does slightly change the Strouhal number values. The type of material used in the housing does alter the frequency slightly for the thick shell. Comparing cases 1 and 11, it is seen that an increase in the value of the Young's modulus causes an increase in the frequency.

The conclusions that can be drawn from the results are as follows. First, the density of the internal fluid can be used most effectively in controlling the natural frequency of the cylindrical shell. This presents a problem to line array designers, however, since the internal fluid is also used to insure neutral buoyancy. Secondly, by decreasing the length of the shell the natural frequency is increased significantly. Thus, the length can be used as an effective frequency control parameter. Finally, thin shells are more influenced by pressure variations than are thick shells. Since line array housings are necessarily thick, pressure variations are not a problem as far as the vibrational characteristics are concerned.

References

[1] Berry, J. and Reissner, E., "The Effect of an Internal Compressible Fluid Column on the Breathing Vibrations of a Thin Pressurized Cylindrical Shell," *Journal of the Aerospace Sciences*, Vol. 25, No. 5, May 1958, pp. 288–294.

[2] Reissner, E., "On Transverse Vibrations of Thin, Shallow, Elastic Shells," *Quarterly of Applied Mathematics*, Vol. 13, 1955, pp. 169–176.

[3] Herman, H. and Klosner, J., "Transient Response of a Periodically Supported Cylindrical Shell Immersed in a Fluid Medium," *Journal of Applied Mechanics*, Sept. 1965, pp. 562–568.

[4] Kraus, H., *Thin Elastic Shells*, Wiley, New York, 1967, pp. 289–302.

[5] McCormick, M., "Axial and Radial Vibrations of a Liquid Filled Cylindrical Shell Submerged in a Heavy Fluid," Rept. 69-R-002, July 1969, Chesapeake Instrument Corp., Shadyside, Md.

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